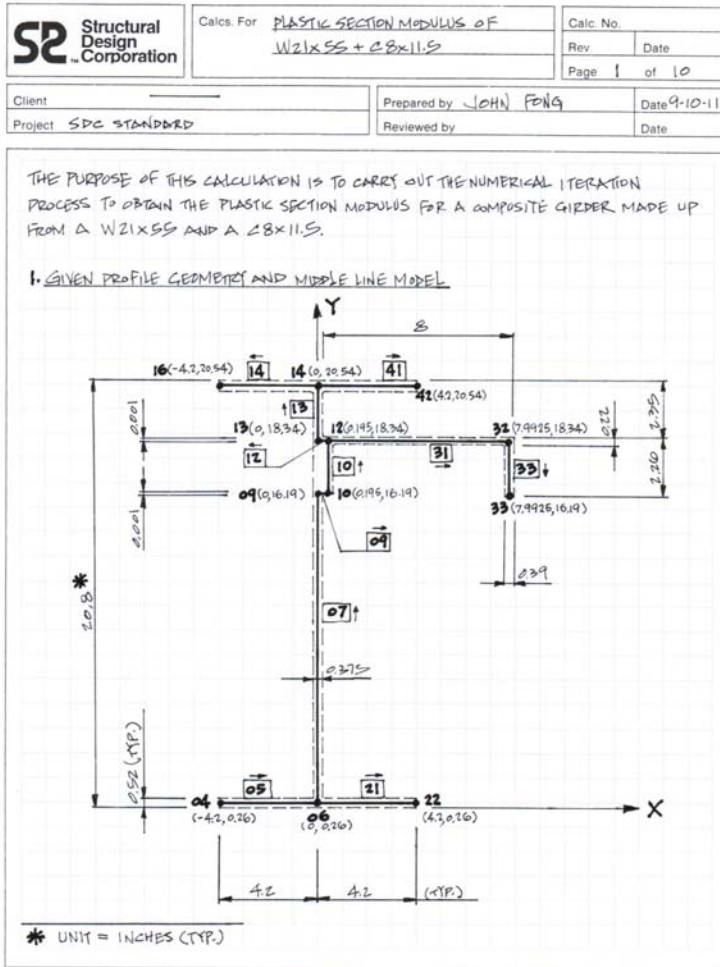


Plastic Section Modulus of Sections with Arbitrary Profile Geometry

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Plastic section modulus is one of the essential properties for steel design per limit states strength criteria. Both LRFD and ASD relate to plastic section modulus. This relevance remains valid whether or not dealing with “plastic design”.



In engineering mechanics, “Plastic” is complement to “Elastic”. These two words characterize the material stress-strain relationship at varied stages. Yet we seem to favor less towards “Plastic”. Phrases such as “Centroid” or “Section Modulus” missing proper qualification seldom relate to “Plastic Centroid” or “Plastic Section Modulus”. However, to avoid confusion, we should address these terms in full: “Elastic Centroid (EC)”, “Elastic Principal Axes (EPA)”, “Elastic Section Modulus (ESM)”, “Plastic Centroid (PC)”, “Plastic Principal Axes (PPA)” and “Plastic Section Modulus (PSM)” where appropriate.

Before computing the PSM for any given section, one must locate its PC and orient the associated PPA. It is easy to do so for symmetrical sections because the PC either shares the same coordinates with the EC or is at a parallel offset from one of the EPA.

AISC “Steel Construction Manual” has already provided PSM value for regular rolled shapes including a number of I-shapes built-up with cap channels. Besides, the fundamental theory behind is rather simple; thereby PSM for symmetrical shapes of other geometry not found in AISC Manual can still be determined with relative ease.

The problem is how to calculate PSM for irregular shapes with arbitrary profile geometry. Lacking an axis of symmetry as reference, attempting close-formed answer to some of the properties for these profiles is extremely challenging (if not impossible). However, the solution leading to PSM

can always be **approximated** by iteration process except that a “**practical** reference” does not appear to exist in the public domain for engineers to tackle profiles having no axis of symmetry.

“Iteration” may be one of the least favored tasks to most of us because it often takes “time” and “luck”. It sounds more horrifying to link up “Iteration” with **Plastic Center** and **Arbitrary Profile Geometry**. But rest assure, the algorithm presented herein should be easy to follow and fun to implement. By being called a “**practical** reference” here, it brings to mind a **comprehensive numerical example** that would:

- (a) explicitly carry out all the **necessary** steps towards locating the **PC**, calculating the **PSM** and the shape factors
- (b) in due process, establish a standard solution procedure applicable to **any thin-walled section** and a methodology suitable for **automation**.

Most of the formulas already existed elsewhere. The purpose of this article is not about explaining the “**why**” or taking credit of that but only to focus on the “**how**” for practical uses. Our only goal here is to challenge ourselves with a “Homework Problem”, through which we apply the algorithm to practice the **PSM** computation for a “composite profile”.

The example profile is made from a W21X55 with a C8X11.5, which is attached off the web of W21 at a few inches below the top flange of W21. The phrase “any thin-walled section”, not being specific, implies that the algorithm would work for any section types: “open”, “closed” or “mixed”.

For easy references, all the calculation (**Calc**) pages prepared manually for this homework problem were scanned in as part of this article. It takes eight (8) steps in the **Calc** to complete the task. Count only to seven (7) if we exclude the “shape factors” at the very end.

As we go through some of the steps (and pages of **Calc**) in the ensuing paragraphs, subjects that appear trivial, self-explanatory or beyond the scope were bypassed for obvious reasons. It pays to draw closer attention to the “**Process**” in problem solving techniques rather than drilling into numerical or typographical details.

Middle Line Profile Model

A hand sketched middle line model is shown on **Calc** page 1. It resembles an ordinary model used in plane frame analysis. However they serve different purposes. While plane frame model is a serious precursor to stiffness matrix, our middle line model is merely an **idealized** graphical description of a profile’s “geometry” and “element connectivity”. **Middle line** suggests that all the framework linkages in the model would always traverse along the **line of mid-thickness** through each and every profile component regardless to the thickness.

Constructing middle line models is fairly straightforward as if drawing linear elements by connecting dots. Take a plain I-beam for instance, it’s ample enough to “idealize” it with only six (6) nodes that depict (or **digitize**) all the participating components, which include two (2) flange elements and one (1) web element.

For any given profile, its **PSM remains the same** whether we derived that value from a simpler or more complex model. In terms of the number of nodes and segments, simple implies “bare minimum” while complex would unnecessarily incite “extra work”. If the application is for computing **PSM** only, be sensible to use simpler model unless obligated to use the same for broader intents. In our open-sectioned example problem, we made up the “middle line profile model” using only 12 “nodes” and 11 “segments”.

User Coordinate Axes System

This is the X-Y Cartesian system chosen by the user to facilitate data collection. Its “system origin” and “axes orientation” were elected entirely per user’s convenience. “User coordinate system” is a “user **assigned** scheme”, which is neither “**EPA**” nor “**PPA**”. Although the **EPA/PPA** attributes could

be “guessed” correctly in simple cases without calculation, but not to be confused that in essence both EPA and PPA were unique to the given profile and were “mathematically verifiable systems”.

For doubly- symmetric sections, there is no geometric distinction between the “Elastic Principal System” and the “Plastic Principal System”. For these sections we often line up the user axes “intuitively” with at least one of the known principal axes or “purposely” intermix the user scheme with the EPA/PPA system because it is convenient. But on the other hand, for sections with no axis of symmetry, “assigning” user axes for modeling purpose would be rather random because at that stage we don’t really know the EC/PC’s whereabouts, let alone EPA/PPA orientation.

As shown on Calc page 1, the user system origin can be “conveniently” located after defining:

- (a) the “Y-axis” to pass through the centerline of the web of W21
- (b) the “X-axis” to line up with the bottom edge of the bottom flange of W21.

Data Input

Notice that it requires only seven (7) discrete entities of information to fully “describe” the profile geometry. There are two (2) basic data categories that were applicable to our PSM example. One of them is for identification (ID) purpose and the other is for quantity or measurement purpose. Referring to Calc page 1, a sample subset of each discrete data object for our model were brought in here as a starter:

- (a) Node identification ID; “04”, “10”, “32”, “42”, etc.
- (b) Nodal x- coordinates; $X_{04} = -4.2$, $X_{10} = 0.195$, $X_{32} = 7.9925$, $X_{42} = 4.2$, etc.
- (c) Nodal y-coordinates; $Y_{04} = 0.26$, $Y_{10} = 16.19$, $Y_{32} = 18.34$, $Y_{42} = 20.54$, etc.
- (d) Segment identification ID (in boxes); “05”, “09”, “33”, “41”, etc.
- (e) Segment “Incidence From” node ID; “04”, “09”, “32”, “14”, etc.

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Project <u>SDC STANDARD</u>	Reviewed by _____	Date _____

2. LOCATION OF ELASTIC CENTROID

t = SEGMENT THICKNESS > 0
 x_i, y_i = USER SYSTEM x, y COORDINATE OF SEGMENT FROM-NODE
 x_j, y_j = USER SYSTEM x, y COORDINATE OF SEGMENT TO-NODE
 L = SEGMENT LENGTH = $[(x_j - x_i)^2 + (y_j - y_i)^2]^{1/2} > 0$
 A = SEGMENT AREA = $[L][t] > 0$
 Ax = FIRST MOMENT ABOUT Y-AXIS = $[A](x_i + x_j) / 2$
 Ay = FIRST MOMENT ABOUT X-AXIS = $[A](y_i + y_j) / 2$

SEGMENT ID	t	COORDINATES				L	A	Ax	Ay
		x_i	y_i	x_j	y_j				
05	0.52	-4.2	0.26	0	0.26	4.2	2.184	-4.5864	0.5618
07	0.375	0	0.26	0	16.19	15.93	5.9738	0	49.1341
09	0.001	0	16.19	0.195	16.19	0.195	0.002	0	0.0032
10	0.765	0.195	16.19	0.195	18.34	2.15	16.447	0.3207	23.3966
12	0.001	0.195	18.34	0	18.34	0.195	0.002	0	0.0036
13	0.375	0	18.34	0	20.54	2.2	0.825	0	16.038
14	0.52	0	20.54	-4.2	20.54	4.2	2.184	-4.5864	44.8594
21	0.52	0	0.26	4.2	0.26	4.2	2.184	4.5864	0.5618
31	0.22	0.195	18.34	7.9925	18.34	7.7975	1.7154	7.0226	31.4614
33	0.39	7.9925	18.34	7.9925	16.19	2.15	0.8385	6.7017	14.4767
41	0.52	0	20.54	4.2	20.54	4.2	2.184	4.5864	44.8594
Σ							19.7338	4.0451	130.3679

$C_x = x_{ELASTIC\ CENTROID} = \Sigma Ax / \Sigma A = 0.7117$
 $C_y = y_{ELASTIC\ CENTROID} = \Sigma Ay / \Sigma A = 11.6737$

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- (f) Segment “Incidence To” node ID; “06“, “10“, “33“, “42“, etc.
- (g) Segment material thickness; 0.52, 0.001, 0.39, 0.52, etc.

In this application: Data for ID purpose are of **symbolic type** while that for nodal coordinates and segment thickness are of **numerical type**. From a “**data design**” viewpoint, information for “**Nodes**” should take higher precedence over that for “**Segments**”. Here are some further details and helpful tips on model data:


- (a) the **numbering** designation and the input sequence for both the “Node” and “Segment” IDs were totally **random**. Symbolic data type offers advantage of using any alphanumeric-text characters as data field content. We are not compelled to identify Nodes and Segments with pure digits as in “04”, “11” or “32”. Any other combination texts such as “A1”, “B2” or “C2901” would do if that serves our descriptive purpose.
- (b) when arranging “segment terminal incidences”, there is no rule over which end should be the “From” node or the “To” node. Notice on Calc page 1, a small arrow was shown along each segment ID box demonstrating how liberal the incidence paths were chosen. Given such diverse freedom in incidence scheme, we should always avoid assigning the same path more than once.
- (c) typical segment should have “uniform thickness” between its “From-node” and “To-node”. To handle changes in thickness within the extents of an element, extra node(s) should be added in between so that each resulting sub-segment would comply with the “uniform thickness rule” on its own.
- (d) material thickness for segments joined by weld or bolt should be **added** into a sum from all the participating segments. For instance: if the web thickness of W21 = 0.375 and flange thickness of C8 = 0.39, then for segment 10, we would use a combined thickness = 0.765 replicating the fact that both W21 web and C8 flange would respond in unison under loads.
- (e) some segments do not really exist but were incorporated into the model only for linkage transition continuity purpose. The thickness of these segments could be regarded as “**insignificant**”. As for segments 9 and 12 in the example: we gave them a fictitious thickness = 0.001 signifying that they are relatively trivial compared to the thickness of other “**significant**” segments in the profile.

Incorporating “insignificant” segments may be crucial in those “All-purpose” models. In certain applications these segments are there for programming purpose to ensure the “logical” continuity of segment linkages. Usually these all-purpose models were intended for calculating the “full suite of section properties” including torsion-related shear-flow logic scheme and warping constant, etc. However, these “insignificant” segments may be omitted from our PSM-specific application if chosen to.

Data Presentation

Engineering document is all about presentation of processes. Arrangement of calculation contents should adhere to styles appealing to the Preparer’s personal liking and be acceptable per Corporate/project standards. But the nature of problem on hand can affect the documentation layouts as well. Among all, there are a handful of familiar styles, each could stand out by several choices of arrangements; in terms of “looks”: (1a) organized in top-down fashion and (1b) structured around results from **data manipulation**; and in terms of “tools” for the respective task: (2a) word processor style, (2b) spreadsheet or **database** style.

Laboring through data manipulation steps can be quite an experience. Some calls for combination of “action queries”, complex “select queries”, cross tabs, aggregate summaries, Boolean operations and computations, etc. Basically it demands series of data “dropping-off” **forwards** and data “looking up” **backwards** in particular for procedures taking up **iteration**.



Calc. For **PLASTIC SECTION MODULUS OF**
W 21 x 55 + C 8 x 11.5

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Prepared by **JOHN FENG**
 Reviewed by _____


Date **9-11-11**
 Date _____

3. ELASTIC PRINCIPAL AXES AND TRANSFORMED COORDINATES

$$I_{xx} = [A][(y_i - C_y)^2 + (y_i - C_y)(y_j - C_y) + (y_j - C_y)^2] / 3$$

$$I_{yy} = [A][(x_i - C_x)^2 + (x_i - C_x)(x_j - C_x) + (x_j - C_x)^2] / 3$$

$$I_{xy} = [A][(x_i - C_x)(y_i - C_y) + (x_j - C_x)(y_j - C_y)] / 3 + [(x_i - C_x)(y_j - C_y) + (x_j - C_x)(y_i - C_y)] / 6$$



SEGMENT ID	SEGMENT MOMENT OF INERTIA			TRANSFORMED COORDINATES			
	I_{xx}	I_{yy}	I_{xy}	x'_i	y'_i	x'_j	y'_j
05	2845177	204768	700897	-4.0744	-11.7387	0.1146	-11.4353
07	1973783	3026	14.663	0.1146	-11.4353	-1.0361	4.453
09	0.004	0.001	-0.0005	-1.0361	4.453	-0.8416	4.4671
10	52.0519	0.4292	-4.7519	-0.8416	4.4671	-0.9969	6.6115
12	0.0087	0.001	-0.0008	-0.9969	6.6115	-1.1914	6.5974
13	50.0924	0.4179	-4.5601	-1.1914	6.5974	-1.3503	8.7917
14	171.6851	20.4768	-54.446	-1.3503	8.7917	-5.5394	8.4883
21	2845177	7.4197	-34.6064	0.1146	-11.4353	4.3037	-11.1319
31	76.2327	28.2132	38.6755	-0.9969	6.6115	6.7802	7.1748
33	26.5363	44.4486	34.1342	6.7802	7.1748	6.9355	5.0304
41	171.6851	7.4197	26.8824	-1.3503	8.7917	2.8387	9.0951
Σ	13147098	132.438	82.079	FROM NODE		TO NODE	

$\alpha = \text{ELASTIC PRINCIPAL AXES ORIENTATION ANGLE W.R.T. USER AXIS} = \frac{1}{2} \tan^{-1} \left(\frac{2 \sum I_{xy}}{\sum I_{yy} - \sum I_{xx}} \right) = -0.0723 \text{ RAD}$

TRANSFORM FROM USER $\begin{Bmatrix} x \\ y \end{Bmatrix}$ INTO PRINCIPAL AXES $\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{Bmatrix} (y - C_y) & (x - C_x) \\ -(x - C_x) & (y - C_y) \end{Bmatrix} \begin{Bmatrix} \sin \alpha \\ \cos \alpha \end{Bmatrix}$

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The solution process for PSM has plenty to do with data manipulation. Although not to the extreme, but it does entail result/summary from series of repetitive sub-procedures. How to transcend data flows from “source” through “transition” to final “target” in a document that ties in all the actions, and how to relay information clearly to the Reviewers (including ourselves) calls for practice and experience.

Data operation catering middle line models can be “node-driven”, “segment-driven” or mixed. Keeping consistency throughout, each individual entity (object) in the data set should be dealt with using consistent algorithms whether logical, numerical or mixed. When executing segment (nodal) data, information related to all 11 segments (12 nodes) would receive the same mass-production treatment. Taking shortcuts in database-intensive applications may be cool and tricky but not

a good practice. In light of the sheer amount of data involved at various stages (steps), it is more practical to present transition data by database-styled tabulations.

Elastic Centroid

It is evident from Calc page 1 that all the “source data” for our model were complete and ready. The question is: Where to start given that the information was all over the page in graphics form? Answer: **Normalize** them using database- (or spreadsheet-) oriented schemes.

How? The answer lies in bringing together both the “nodal coordinates” and the “pertinent connectivity **relationship**” under one single “listing”. The information needed for such listing can be compiled manually from several tables or as result of simple database query. Not only for viewing convenience but more importantly this also helps avoiding “data referential integrity” traps further down the road.

Most engineers know how to **consolidate** “design input” at the beginning of a spreadsheet (or hand Calc) but rarely sink in the idea that this is a part of basic “**data normalization**” effort. What is data normalization? Phrase it in database terms: it is the practice of **analyzing** data structures, establishing data **relationships** and enforcing **singular** data definition (free from double data

entries) but allowing **multiple** references and **protecting** referential integrity. Applying some of that here at the beginning of this calculation, our immediate chores would be to:

- (a) list (sort) the segment **IDs** in whichever sequence that makes the best sense to the task. Form habits to be consistent and maintain standard sort order (ascending or descending) throughout all the tabulations that would appear in the subsequent steps,
- (b) retrieve the coordinates of the starting (From) and ending (To) nodes for each segment.

Calc page 2 shows the “source data” being consolidated with segment ID appearing in the first column. While collecting the data we imposed indices “i” and “j” to each pair of “From node” and “To node” coordinates. Identified below are the definitions of data under columns 2 through 6:

[t] = segment thickness

[x_i], [y_i] = segment “From Node” X, Y coordinate based on user system

[x_j], [y_j] = segment “To Node” X, Y coordinate based on user system

See the word “segment” leads in every definition statement above? It indicated that both the “node-based **physical** information” and the “segment-based **logical** connectivity” have been coerced (**merged**) into a multi-columned segment-based data set.

It also hinted that following this step the “Node” entity would be “logically” done with. Filling in its place would only be [x_i], [y_i], [x_j] and [y_j]. Here is a further implication: all the subsequent data manipulation steps yet to be applied would only be “segment-oriented”.

Finishing the tabulation columns 7 through 10 on **Calc** page 2 were the elastic centroid-related “transition data”. They were calculated for each segment by these definitions:

[L] = segment length = $[(x_j - x_i)^2 + (y_j - y_i)^2]^{1/2}$

[A] = segment area = [L] [t]

[Ax] = first moment of segment area about the user Y-axis = [A] (x_i + x_j) / 2

[Ay] = first moment of segment area about the user X-axis = [A] (y_i + y_j) / 2

Once summing up the individual [A], [Ax] and [Ay] respectively into [ΣA], [ΣAx] and [ΣAy], the coordinates of **EC** could be determined as:

[cx] = X coordinate of **EC** based on user system = ΣAx / ΣA

[cy] = Y coordinate of **EC** based on user system = ΣAy / ΣA

Elastic Principal Axes

After locating the **EC**, we can now pre-position the **EPA** by creating a **temporary** Cartesian [X_c-Y_c] system with these features:

- (1) its origin to pass through the **EC** and
- (2) its axes orientation to match that of the user [X-Y] axes (see sketch on **Calc** page 3).

The only chores left for finalizing the **EPA** is the axis orientation that can be obtained by way of moment of inertia. The individual segment moment of inertia [I_{xx}], [I_{yy}] and [I_{xy}] with respect to (w.r.t.) the [X_c-Y_c] system were listed under columns 2 through 4 on **Calc** page 3 where:

[I_{xx}] = [A] [(y_i - cy)² + (y_i - cy)(y_j - cy) + (y_j - cy)²] / 3

[I_{yy}] = [A] [(x_i - cx)² + (x_i - cx)(x_j - cx) + (x_j - cx)²] / 3

$$[I_{xy}] = [A] \{ [(x_i - cx)(y_i - cy) + (x_j - cx)(y_j - cy)] / 3 + [(x_i - cx)(y_j - cy) + (x_j - cx)(y_i - cy)] / 6 \}$$

Applying an algebraic summation over the individual segment moment of inertia $[I_{xx}]$, $[I_{yy}]$ and $[I_{xy}]$ would lead to $[\Sigma I_{xx}]$, $[\Sigma I_{yy}]$ and $[\Sigma I_{xy}]$, in that order. The orientation angle $[\alpha]$ of the elastic principal axes w.r.t. the user system would follow:

$$[\alpha] = 0.5 * \tan^{-1}[2 \Sigma I_{xy} / (\Sigma I_{yy} - \Sigma I_{xx})]$$

For symmetrical sections, the product moment of inertia $[I_{xy}] = 0$ for all the segments. Therefore $[\alpha] = 0$ in that case, otherwise the angular measured in radians for $[\alpha]$ is positive counterclockwise (and negative clockwise).

The EPA Cartesian system is designated with superscript into $[X']$ and $[Y']$ to differentiate it from: (1) the user-sourced $[X-Y]$ system and (2) the pre-positioned $[X_c-Y_c]$ system. The $[X']$ axis, drawn through the EC at an acute angle $[\alpha]$ counterclockwise from the user-X axis, is the Elastic Principal Strong Axis whereas the $[Y']$ axis through EC at 90 degrees from $[X']$ becomes the Elastic Principal Weak Axis.

Coordinate Transformation

Not all the geometric property calculations require Coordinate Transformation. The need for transformation depends on a few factors:

- (1) the geometric feature of the profile, (2) the section property of interest, and most importantly (3) the relationship between the EPA and the reference coordinate system.

For doubly- symmetric sections, the (physical locations) coordinates of EC, PC and the shear center (or the center of twist) are identical. If we take advantage of the symmetry by: (1) assigning the user system origin to coincide with the EC and (2) matching the orientations of user-axis along that of the EPA, then transformation of coordinates would be unnecessary when referring to any point of interest because the user-assigned system is physically the same as the EPA system.

For mono- symmetric sections, one of the EPA axes always divides the profile into two symmetrical halves. If we line up one of the user-axes with the axis of symmetry, then relating any point of

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4 PLASTIC PRINCIPAL AXES STRATEGY

$\begin{Bmatrix} e_x \\ e_y \end{Bmatrix}$ = PARALLEL SHIFT FROM $\begin{Bmatrix} Y' \\ X' \end{Bmatrix}$

$\begin{Bmatrix} x_{pt} \\ y_{pt} \end{Bmatrix}$ = $\begin{Bmatrix} X' \\ Y' \end{Bmatrix}$ BASED SEGMENT [FROM NODE TO NODE] COORDINATES SHIFT BY $\begin{Bmatrix} e_x \\ e_y \end{Bmatrix}$

= $\begin{Bmatrix} x'_i - e_x, y'_i - e_y \\ x'_j - e_x, y'_j - e_y \end{Bmatrix}$

$\begin{Bmatrix} X_p \\ Y_p \end{Bmatrix}$ = COORDINATE SYSTEM AFTER SHIFTING BY $\begin{Bmatrix} e_x \\ e_y \end{Bmatrix}$ FROM $\begin{Bmatrix} X' \\ Y' \end{Bmatrix}$

$\begin{Bmatrix} A_{xp} \\ A_{xn} \\ A_{yp} \\ A_{yn} \end{Bmatrix}$ = SEGMENT AREA IN $\begin{Bmatrix} +X_p \\ -X_p \\ +Y_p \\ -Y_p \end{Bmatrix}$ ZONE

IDEAL CONDITION IS ACHIEVED WHEN SHIFTING $\{e_x, e_y\}$ FROM $\{Y', X'\}$ TO THE LOCATION OF PLASTIC PRINCIPAL AXES $\{Y_p, X_p\}$ THAT

$\Sigma A_{xp} = \Sigma A_{xn} = \Sigma A_{yp} = \Sigma A_{yn}$

= $\Sigma A / 2$

= $19.7338 / 2 = 9.8669 = A_{IDEAL}$

AXIS SHIFTS

BENDING ABOUT X_p

BENDING ABOUT Y_p

interest from the user sourced system w.r.t. EPA would need to deal with [cx] or [cy] only because the EPA orientation angle $[\alpha] = 0$.

For unsymmetrical sections, EC, PC and shear center are **distinctly positioned**. Establishing user system for that would take some wild guessing. It may be probable, but vary unlikely to be able to pin down the user-system origin right over the EC without knowing where it's at in advance. Let alone matching up axes orientation with the EPA's. Therefore unless we nailed the user system origin dead on the EC and lined up our axes with EPA, otherwise the user system-based coordinates at any point of interest must be transformed w.r.t. EPA before any serious engineering calculation can begin.

Thanks to the non-zero axis offsets [cx], [cy] and the orientation angle $[\alpha]$. Some of us accustomed to the friendliness of symmetric sections may find it difficult when switching to a cumbersome EPA-based environment. A generalized coordinate transformation procedure for unsymmetrical sections involves both **translation** and **rotation**. Translation is necessary when user system origin is at non-zero offsets [cx] and/or [cy] from the EC while the rotation portion is needed when the EPA orientation angle $[\alpha] \neq 0$.

Combining the translation of coordinate through parallel displacements with axes rotation, we arrive at the $\{X', Y'\}$ coordinates transformed into the EPA system:

$$[X'] = (y - cy) * \sin\alpha + (x - cx) * \cos\alpha$$

$$[Y'] = -(x - cx) * \sin\alpha + (y - cy) * \cos\alpha$$

It is important to recognize that EPA system is the **core** of unsymmetrical sections. It governs everything from nodal coordinates, elastic section properties, global structural response to all aspects of internal stress calculations, etc. In the world of unsymmetrical sections, not only all the references to geometric dimension had to undergo coordinate transformation but also the formulas for all the common design functions would end up much more complex. All of that owe to the coordinate transformation that played a significant role in "true structural engineering" that has anything to do with unsymmetrical sections.

Upon substituting the "From-node"/"To-node" coordinates $[x_i]$, $[y_i]/[x_j]$, $[y_j]$ (Calc page 2) into the matching formulas for $[X']$ and $[Y']$, the resulting transformed coordinates $[x'_i]$, $[y'_i]/[x'_j]$, $[y'_j]$ were shown under columns 5 through 8 on Calc page 3.

Now a timely question: What if we "**forgot**" to perform coordinate transformation when dealing with unsymmetrical sections (in other words: failed to acknowledge the implication from elastic principal axes)?

The answer depends on what is at stake but the fact is: we could still plug in the **erroneous** data and the **incorrect** section properties into any set of equations (formulas) for all the familiar purposes and can't tell the difference if we erred. That is a typical "garbage in garbage out" situation as "everything right had been seriously wrong". This could be bad news for some of the old-fashioned crane runway girders inherent with unusual section geometry but qualified with structural properties and/or assumptions that may be wrong.

How do we "know" we have a problem? Well, it's difficult to "know" it. But we may **actually** "see" the structural distress based on the deficiency findings tallied up from close-up in-service inspections. Otherwise at least **numerically** we could look for some of the "obvious" such as: numbers that shouldn't be zero, trivial, infinitive or negative and what not, PC coordinates ended up at the least anticipated location or the value of PSM had been found less than that of ESM, etc.

Transition from Elastic to Plastic

EPA and PPA are sibling neutral-axis systems closely related to bending. A typical similarity shared between them is that each component axis from either system always demarcates the section

profile into a compression zone and a tension zone. The only difference is that each axis would attend its unique and appropriate way of “rationalizing” the zone boundaries.

Before the **extreme fiber stress** of a section reaches “specified minimum yield stress” (F_y) under bending, EPA always takes full control and the stress distribution diagram is always of triangular shape. The section stays in “elastic” so long as the stress does not exceed F_y . Notably the stress-strain relationship in both the compression and the tension zones is always linear. Besides that, the EC remains fixed in location.

After **extreme fiber stress** peaks at F_y , the **extreme fiber strain** level can still grow until the section fails or becomes “fully plastic”. In transition from elastic to plastic as the **extreme fiber strain** increases, the rate of “change” in stress distribution in each zone (tension or compression) could be at the same or at a diverse **gradient** depending on these factors:

- the setup of profile component geometry in the tension zone and that in the compression zone
- the material stress-strain behaviors (relationships) under tension and that under compression

Through stages, the “historical” shapes of stress distribution would initiate from “triangular” into “trapezoidal” then finally “rectangular” (block). Correspondingly, the section would toil through phases from “elastic” into “elastic-plastic” and then “full plastic”.

Maintaining the sum of forces in the tension zone (ΣF_t) and the sum of forces in the compression zone (ΣF_c) in equilibrium at all stages is the key to solving PSM. This simply requires that $\Sigma F_t = \Sigma F_c$ at all times.

While the strain keeps on increasing after the stress level already peaked at F_y , the EC would have to **move** continually to go along with the changes in the (trapezoidal) shape of stress distribution before reaching the full plastic stage. Along with each finite movement of EC, the EPA would evolve (at parallel offsets from their instantaneous locality) into a new position only where mathematically $\Sigma F_t = \Sigma F_c$ holds true.

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$\{K_{tx}\}$ = KEYWORD { FULL, PART, HALF, NONE } APPLIED TO PORTIONING OF SEGMENT AREA WITHIN $\{+X_p\}$ ZONE
 $\{R_{tx}\}$ = PROJECTION OF NORMALIZED $\{X_p\}$ AXIS, APPLICABLE ONLY TO $\{K_{tx}\} = \{PART\}$, SEGMENT AREA ONTO THE $\{Y_p\}$ AXIS
 $\{R_{ty}\}$ = PROJECTION OF NORMALIZED $\{Y_p\}$ AXIS, APPLICABLE ONLY TO $\{K_{ty}\} = \{PART\}$, SEGMENT AREA ONTO THE $\{X_p\}$ AXIS
 A_{inc} = PORTION OF SEGMENT AREA INCLUSIVE TO $\{A_{xp}$ OR $A_{yp}\}$
 $=$ PRODUCT OF $A \times \begin{cases} 1 \\ R_{tx} \text{ OR } R_{ty} \\ 0.5 \\ 0 \end{cases}$ FOR $\{K_{tx}$ OR $K_{ty}\} = \begin{cases} FULL \\ PART \\ HALF \\ NONE \end{cases}$

5. ITERATION FOR X_p AXIS * PART = PARTIAL

LET CURRENT $e_y = 0$, CYCLE = 1

SEGMENT ID	y_{pf}	y_{pt}	A	K_{ty}	R_{ty}	A_{inc}
05	-11.7387	-11.4353	2.184	NONE	-	0
07	-11.4353	4.453	5.9738	PART	0.376	1.6743
09	4.453	4.4671	0.0002	FULL	-	0.0002
10	4.4671	6.6115	1.6447	FULL	-	1.6743
12	6.6115	6.5974	0.0002	FULL	-	0.0002
13	6.5974	8.7917	0.825	FULL	-	0.825
14	8.7917	8.4883	2.184	FULL	-	2.184
21	-11.4353	-11.1319	2.184	NONE	-	0
31	6.6115	7.1748	1.7154	FULL	-	1.7154
33	7.1748	5.0304	0.8385	FULL	-	0.8385
41	8.7917	9.0951	2.184	FULL	-	2.184

SUMMARY CYCLE 1

$\Sigma A_{inc} = 11.0664$

$A_{IDEAL} = 9.8669$ (-)

$A_{err} = 1.1995$

$\frac{A_{err}}{A_{IDEAL}} = 0.1216$

$>> TRIVIA$

$\Sigma R_{ty} = 0.376$

NEXT $e_y = CURR e_y + 0.5 \cdot \frac{A_{err}}{\Sigma R_{ty}}$

$= 0 + \frac{1.1995}{0.376} \times 0.5$

$= 1.5951$

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Finally at the full plastic phase, both **EC** and **EPA** would stop moving and shifting. The concluding geometric departure from the original **EC/EPA** position would become the **PC/PPA**, which is shown qualitatively as the “**AXIS SHIFTS**” on Calc page 4.

Iterations and Iteration Strategy

Prior to concluding **EC/EPA**’s progression into **PC/PPA** while **EC** is still in transit, at each “**incremental axis shifts**” the exact coordinates of the “dynamic” position of **EC**, with respect to the $[X'-Y']$ system, can always be pinned down **intermittently**. Meanwhile all the segment terminal incidences affected by such “axis shift” can be instantaneously transformed w.r.t. $[X'-Y']$ based system as well.

As a result, each “**incremental axis shift**” can be considered as the **trigger** to a new “iteration cycle”. The objectives of tracking down: (1) the changes in coordinate geometry at each segment terminals and (2) the numerical side effects imposed upon other entities at any given “cycle” constitutes our “Iteration Strategy”.

Basically our strategy can be deduced into two primary tasks: (1) to calculate the **transient** coordinates and (2) to maintain the **transient** forces in equilibrium. Understandably both of these tasks have to deal with **frequent** information storing and retrieving during each iteration cycle.

To appreciate the importance and the dynamics of coordinate transformation of segment terminal incidences, let us begin by:

- (a) choosing a nodal point, which may be the “from-node” or the “to-node” of a segment, given their transformed coordinate pairs; $\{x'_i, y'_i\}$ or $\{x'_j, y'_j\}$
- (b) considering $[e_x]$ and $[e_y]$ as the projected lengths onto any Cartesian X-Y axes independent from any specific system.

The wild idea is somehow we should be able to derive the **PC** through these afore-mentioned entities. Instead of treating $[e_x]$ and $[e_y]$ as some random sidetracks, why not we tie them in with the known **EPA** by the following characteristics:

$\{e_x, e_y\}$ = parallel offset dimensions from **EPA** X', Y'

$\{X_p, Y_p\}$ = new reference axes system based on e_y, e_x offsets from X', Y'

It is plain to see from these definitions that $[X_p]$ and $[Y_p]$ axes are correspondingly parallel to $[X']$ and $[Y']$. Our imaginary segment’s “from-node coordinates”, if designated as $[X_{pf}]$ and $[Y_{pf}]$ with reference to the offset axes $[X_p]$ and $[Y_p]$, can now be calculated as $X_{pf} = (x'_i - e_x)$ and $Y_{pf} = (y'_i - e_y)$. Accordingly the “to-node coordinates”, designated as $[X_{pt}]$ and $[Y_{pt}]$, can be expressed as $(x'_j - e_x)$ and $(y'_j - e_y)$. To write these coordinates using more structured statements:

$\{X_{pf}, Y_{pf}\} = X_p - Y_p$ based segment “**from-node**” coordinate = $\{x'_i - e_x, y'_i - e_y\}$

$\{X_{pt}, Y_{pt}\} = X_p - Y_p$ based segment “**to-node**” coordinate = $\{x'_j - e_x, y'_j - e_y\}$

All we did so far was nothing but “applying a simple coordinate **translation**” from the **EPA** by some respective quantities $[e_x]$ and $[e_y]$. It is important to realize for the moment that the **translation** is against the **EPA** but not the user $[X-Y]$ axes. To think beyond $[X']$ and $[Y']$ as in pure elastic stage, the introduction of $[X_p]$ and $[Y_p]$ now point us into the direction of elastic-plastic and full plastic stages. In order to deal with $\{X_{pt}, Y_{pt}\}$ in a more direct way **numerically**, by some means we need to skip the intermediate $\{x'_j, y'_j\}$ and yet without missing any of the key players: $[c_x]$, $[c_y]$, $[\alpha]$, $[e_x]$ and $[e_y]$.

If we redefine these transient coordinates using long-hand expansions that incorporate coordinate references “rooting back” all the way to the original user-sourced $[X]$, $[Y]$ system, then finally we arrive at these generalized expressions as the segment incidence coordinates:

$$\{X_{pf}, Y_{pf}\} = \{(y_i - cy) * \sin\alpha + (x_i - cx) * \cos\alpha - e_x, -(x_i - cx) * \sin\alpha + (y_i - cy) * \cos\alpha - e_y\}$$

$$\{X_{pt}, Y_{pt}\} = \{(y_j - cy) * \sin\alpha + (x_j - cx) * \cos\alpha - e_x, -(x_j - cx) * \sin\alpha + (y_j - cy) * \cos\alpha - e_y\}$$

With good handle on the coordinate geometry using these formulas, we could then properly evaluate the force equilibrium condition engaging both the tension and the compression zones. The boundary in between zones were already marked clearly by $[X_p]$ and $[Y_p]$ axis, which in turn were pre-positioned at $[e_y]$ and $[e_x]$ offsets from the EPA.

The formula of force equilibrium at plastic stage, $[\Sigma F_t] = [\Sigma F_c]$, can be expanded in terms of “stress multiplied by area”, i.e. $[F_y] * [\Sigma A_t] = [F_y] * [\Sigma A_c]$ where $[\Sigma A_t]$ and $[\Sigma A_c]$ are the sum of individual segment area prescribed by $[X_p]$ and $[Y_p]$ axes into respective tension and compression zones.

Upon canceling the $[F_y]$ term, our final equilibrium requirement now becomes $[\Sigma A_t] = [\Sigma A_c]$. In other words: “force equilibrium” has turned into “area equilibrium”. From the constant $[\Sigma A]$ consisting the sum of $[\Sigma A_t] + [\Sigma A_c]$, it leads to the simple condition of:

$$[\Sigma A_t] = [\Sigma A_c] = [\Sigma A] / 2$$

In actual practice, the generic equality $[\Sigma A_t] = [\Sigma A_c]$ should always be assessed twice: once for the $[X_p]$ axis bending and once for the $[Y_p]$ axis bending.

Since we don't know, for either axis of bending, which zone is compression and which is tension, then why not categorizing these areas based on the orientation sense of each individual axis. For example, if we define a subscript “ x_p ” to stand for “X-positive” and “ x_n ” for “X-negative”, then the area term $[A_{xp}]$ would represent the segment area contained within the $[+X]$ quadrants. Likewise $[A_{yn}]$ would be the segment area within the $[-Y]$ quadrants.

For the convenience of further explanation: we let $[A_{xp}]$, $[A_{xn}]$, $[A_{yp}]$ and $[A_{yn}]$ be the portion of “segment area” allotted into each respective zone (quadrants); $[+X_p]$, $[-X_p]$, $[+Y_p]$ and $[-Y_p]$. Once the “from-node” coordinates $\{X_{pf}, Y_{pf}\}$ and the “to-node” coordinates $\{X_{pt}, Y_{pt}\}$ for the segment of

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SEGMENT ID	y_{pf}	y_{pt}	Δ	K_{xy}	R_{xy}	A_{inc}	
05	-12.3338	-12.0304	2.184	NONE	-	0	SUMMARY CYCLE 2 $\Sigma A_{inc} = 124666$ $A_{ideal} = 9.8669 (-)$ $A_{err} = 0.5997$ $\frac{A_{err}}{A_{ideal}} = 0.0608$ $> TRIVIA$ $\Sigma R_{xy} = 0.376$ NEXT $e_y = CURR e_y + 0.5 \cdot \frac{A_{err}}{\Sigma R_{xy}}$ $= 1.5951 + \frac{0.5997}{0.376} \times 0.5$ $= 2.3926$
07	-12.0304	2.858	5.9738	PART	0.376	1.0745	
09	2.858	2.8721	0.0002	FULL	-	0.0002	
10	2.8721	5.0164	1.6447	FULL	-	1.6447	
12	5.0164	5.0024	0.0002	FULL	-	0.0002	
13	5.0024	7.1966	0.825	FULL	-	0.825	
14	7.1966	6.8932	2.184	FULL	-	2.184	
21	-12.0304	-12.727	2.184	NONE	-	0	
31	5.0164	5.5797	1.7154	FULL	-	1.7154	
33	5.5797	3.4353	0.8385	FULL	-	0.8385	
41	7.1966	7.5	2.184	FULL	-	2.184	

SUMMARY OF FURTHER ITERATIONS							
CYCLE	CURR e_y	ΣA_{inc}	A_{err}	$\frac{A_{err}}{A_{ideal}}$	ΣR_{xy}	$0.5 \frac{A_{err}}{\Sigma R_{xy}}$	NEXT e_y
2	2.3926	10.1668	0.2999	0.0304	0.376	0.3988	2.7914
4	2.7914	10.0169	0.15	0.0192	↑	0.1995	2.9908
5	2.9908	9.9419	0.075	0.0076	↑	0.0997	3.0904
6	3.0904	9.9044	0.0375	0.0038	↑	0.0499	3.1403
7	3.1403	9.8857	0.0188	0.00191	↑	0.025	3.1652
8	3.1652	9.8763	0.0094	0.00095	0.376	0.0125	3.1777

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interest are known, its respective area $[A_{xp}]$, $[A_{xn}]$, $[A_{yp}]$, $[A_{yn}]$ dispersed into each corresponding zone can be figured out from that.

Through trial and error, our ultimate goal becomes locating the PC with **desirable accuracy** at certain offset distances, $[e_x]$ and $[e_y]$, from the EC. In a nutshell the iteration strategy would involve these steps:

- (a) initialize reasonable values for e_x and e_y
- (b) based on e_x and e_y , perform a generalized coordinate transformation and obtain values of X_{pf} , Y_{pf} , X_{pt} , Y_{pt} for each segment
- (c) calculate individual **segment area dissect** in each applicable $[+X_p]$, $[-X_p]$, $[+Y_p]$, $[-Y_p]$ zone as A_{xp} , A_{xn} , A_{yp} , A_{yn}
- (d) sum all the individual zone areas into ΣA_{xp} , ΣA_{xn} , ΣA_{yp} , ΣA_{yn}
- (e) evaluate for ideal condition whether: $\Sigma A_{xp} = \Sigma A_{xn} = \Sigma A_{yp} = \Sigma A_{yn} = \Sigma A / 2$
- (f) stop if condition is (e) is true, otherwise adjust the value of $[e_x]$ and/or $[e_y]$, continue from steps (b) through (f) until ideal condition in (e) is met.

To avoid confusion for now, we provided only skeleton details as needed without stuffing too much in each step until we delve into the subjects of “Segment Area Strategy” and “Convergence Strategy”. After all it would be much easier in the actual iteration process as we follow the sample Calc.

Segment Area Strategy

It is easy to figure out the **full area** $[A]$ of any segment regardless to which coordinate system is in focus. But calculating the **partial** segment area dissects in the tension and compression zones based on the **transient** X_p - Y_p system could be confusing and tricky at times.


Consider any general segment for example, its incidence coordinates based on the $[X_p$ - $Y_p]$ system, $\{X_{pf}, Y_{pf}\}$ and $\{X_{pt}, Y_{pt}\}$, could fall **anywhere** within one or more of these zones $[+X_p]$, $[-X_p]$, $[+Y_p]$ and $[-Y_p]$. Consequently which portion of the segment is to be included into (or excluded from) the matching segment area dissect, whether A_{xp} , A_{xn} , A_{yp} or A_{yn} , could be ambiguous if we haven't specify “which axis of bending is of interest”. What we need now is some rational ways to: (1) organize the segment area dissect attributes **logically** and (2) handle the changes in segment data **efficiently**.

Let us look at the “segment area dissect attributes” based on all the probable zone(s) that may apply to a **thin-walled** segment, then in a general sense, the “area layout/spread” of any segment could be “**True**” to only one of the listed conditions (a) through (f) as follow, with (g) and (h) being the special cases:

- (a) entirely in $[+X_p]$,
- (b) partial $[+X_p]$ and partial $[-X_p]$,
- (c) entirely in $[-X_p]$,
- (d) entirely in $[+Y_p]$,
- (e) partial $[+Y_p]$ and partial $[-Y_p]$,
- (f) entirely in $[-Y_p]$,
- (g) $X_{pf} = X_{pt} = 0$ (special condition),

(h) $Y_{pf} = Y_{pt} = 0$ (special condition).

Now from a different viewpoint: if we assess the “segment area-dissect attributes” based only on the respective axis of bending, then the full area of any segment could be broken down into one of these categories:



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6. ITERATION FOR Y_p AXIS
LET CURRENT $e_x = 0$, CYCLE = 1

SEGMENT NO.	X_{pf}	X_{pt}	Δ	K_{rx}	R_{rx}	A_{inc}
05	-4.0744	0.1146	2.184	PART	0.5214	0.0598
07	0.1146	-1.0361	5.9738	PART	5.1912	0.5951
09	-1.0361	-0.8416	0.0002	NONE	-	0
10	-0.8416	-0.9969	1.6447	NONE	-	0
12	-0.9969	-1.1914	0.0002	NONE	-	0
13	-1.1914	-1.3503	0.825	NONE	-	0
14	-1.3503	-5.5394	2.184	NONE	-	0
21	0.1146	4.3037	2.184	FULL	-	2.184
31	-0.9969	6.7802	1.7154	PART	0.2206	1.4956
33	6.7802	6.9355	0.8385	FULL	-	0.8385
41	-1.3503	2.8337	2.184	PART	0.5214	1.48

SUMMARY CYCLE 1

$\sum \Delta_{inc} = 6.6529$
 $A_{ideal} = 9.8669 (-)$
 $A_{err} = -3.214$

$\frac{A_{err}}{\Delta_{ideal}} = +0.3257$
 \gg TRIVIA
 $\sum R_{rx} = 6.4545$

NEXT $e_x = CURR C_x + 0.5 \frac{A_{err}}{\sum R_{rx}}$
 $= 0 + 0.5 \left(\frac{-3.214}{6.4545} \right)$
 $= -0.24897$

SUMMARY OF FURTHER ITERATIONS

CYCLE	CURR e_x	$\sum A_{inc}$	A_{err}	$\frac{A_{err}}{A_{ideal}}$	$\sum R_{rx}$	$0.5 \frac{A_{err}}{\sum R_{rx}}$	NEXT e_x
2	-0.24897	8.2599	-1.607	-0.1629	6.4545	-0.1245	-0.3735
3	-0.3735	9.0634	-0.8035	-0.0814	6.4545	-0.0622	-0.4357
4	-0.4357	9.4652	-0.4017	-0.0407	6.4545	-0.0311	-0.4668
5	-0.4668	9.669	-0.2009	-0.02036	6.4545	-0.0156	-0.4824
6	-0.4824	9.7665	-0.1004	-0.0102	6.4545	-0.0077	-0.49017

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(a) for Y_p -axis bending:
 A_{xp}, A_{xn}

(b) for X_p -axis bending:
 A_{yp}, A_{yn}

To comprehend all of these with no visual help seems awfully confusing. But they can easily be deciphered from the figures (**BENDING ABOUT X_p** and **BENDING ABOUT Y_p**) shown at the bottom of Calc page 4.

Next we need to: (1) examine the “area dissect aspect” of each segment for either axis of bending, and (2) segregate the outcomes based on how did it participate in the “positive” or “negative” zone (quadrants). Consider this for simplification: no matter which “zone” the segment “area” may “fall into” or be “cut apart” by whichever neutral axis of bending, there are basically only four groups of “area dissect” we have interests in:

“FULL”, “PARTIAL”, “NONE” and “HALF”.

Doing it symbolically by incorporating these abstract terms into our strategy, let us look into only the positive (+) quadrants, for example, and classify the area “participation” (A_+) of each segment into one of these categories:

- (a) A_{+FULL} : segment fits entirely (100%) in the “positive” zone
- (b) A_{+PART} : for segment straddling both zones, only **PARTIAL** area fits in “positive” zone
- (c) A_{+NONE} : not participating (0%) in the “positive” zone for the entire segment fits 100% in the “negative” zone only
- (d) A_{+HALF} : only half (50%) of the area participates in the “positive” zone. This special condition applies only when either $Y_{pf} = Y_{pt} = 0$ or $X_{pf} = X_{pt} = 0$

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By the same token we can define the counterpart area terms, $[A_{-FULL}]$, $[A_{-PART}]$, $[A_{-NONE}]$ and $[A_{-HALF}]$, for the “negative“ zone (quadrants). But we really have no need of doing so. Remember: we are looking for “area equilibrium” between the “+” and “-“ zones, and so for bending about the X_p -axis we may take advantage to retain only the area $[A_{YP}]$ that lies in the “+Y” zone and leave off the calculation for $[A_{YN}]$ all together. (This would cut down the amount of work by half and simplify our iteration process drastically.)

On top of that, to figure out the positive zone area $[A_{+FULL}]$, $[A_{+PART}]$, $[A_{+NONE}]$ or $[A_{+HALF}]$ due to bending about the X_p -axis, we only need to focus just the effect due to coordinates $[Y_{pf}]$ and $[Y_{pt}]$ (and leave off any contribution from $[X_{pf}]$ and $[X_{pt}]$ all together that would cut down our workload even further).

Now we are all set for the hands-on **Segment Area Strategy**. As an example we would look into the procedure related to X_p -axis bending. It is necessary to declare these initial conditions at the beginning:

- (1) let $[e_y] = e_{init}$, (an initial value)
- (2) let $[A_{+FULL}] = 0$. Herein the term $[A_{+FULL}]$ has been generalized to also include $[A_{+HALF}]$.
- (3) let $[A_{+PART}] = 0$.

Next we construct the **pseudo codes**, for a typical segment that work something like this:

Let the “full area” of the current segment = A
 Retrieve the coordinates Y_{pf} and Y_{pt} (by coordinate transformation based on offset e_y)

Start evaluating coordinate Y_{pf}

$$Y_{pf} < 0$$

If only $Y_{pt} > 0$ then
 $A_{+PART} = A_{+PART} + Y_{pt} * A / (Y_{pt} - Y_{pf})$

$$Y_{pf} = 0$$

If only $Y_{pt} > 0$ then
 $A_{+FULL} = A_{+FULL} + A$

If only $Y_{pt} = 0$ then
 $A_{+FULL} = A_{+FULL} + A / 2$

$$Y_{pf} > 0$$

If only $Y_{pt} \geq 0$ then
 $A_{+FULL} = A_{+FULL} + A$

If only $Y_{pt} < 0$ then
 $A_{+PART} = A_{+PART} + Y_{pf} * A / (Y_{pf} - Y_{pt})$

End of Y_{pf} evaluation

This should be applied to each and every segment. At the end, our final $[A_{+FULL}]$ and $[A_{+PART}]$ would have been summed into $[\Sigma A_{+FULL}]$ and $[\Sigma A_{+PART}]$, respectively. With $[\Sigma A_{YP}]$ being the total area in the $[+Y_p]$ zone we now have:


$$\Sigma A_{YP} = \Sigma A_{+FULL} + \Sigma A_{+PART}$$

If an equilibrium condition is reached at this point then the area sum must also satisfy the equality: $[\Sigma A_{YP} = \Sigma A / 2]$. But chances are that this may be wishful thinking at the very first try. In other words, based on the guesstimated $[e_{init}]$ we come to a $[\Sigma A_{YP}]$ value that did not pass the equilibrium check. More specifically: $[\Sigma A_{YP} <> \Sigma A / 2]$.

To move forward we ought to make another educated guess of $[e_y]$. For that we could work out the numerical value later but first let us introduce another symbol: $[e_{yincr}] =$ **incremental adjustment** to

the previously estimated $[e_y]$ offset from X_p -axis. With $[e_{yincr}]$ still an unknown for the time being, we can certainly hope for these two perfect conditions:

- (1) $[e_{yincr}]$ is indeed the final incremental shift we were seeking and
- (2) all the current $[A_{+FULL}]$ stays being "FULL" and all the current $[A_{+PART}]$ remains being "PARTIAL"



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7. PLASTIC SECTION MODULUS

$\begin{Bmatrix} e_y \\ e_x \end{Bmatrix}$ = LOCATION OF PRINCIPAL PLASTIC AXIS $\begin{Bmatrix} X_p \\ Y_p \end{Bmatrix}$ AS OFFSET FROM $\begin{Bmatrix} X' \\ Y' \end{Bmatrix}$
 $= \begin{Bmatrix} 3.19 \\ -0.4979 \end{Bmatrix}$ BOTH REACHING CONVERGENCE AT ITERATION CYCLE = 10

$\{Z_x, Z_y\}$ = PLASTIC SECTION MODULUS
 = FIRST MOMENT OF SEGMENT AREA ABOUT $\{X_p, Y_p\}$
 $= \frac{1}{2} \left\{ R+y(Y_{pf}^2+Y_{pt}^2), R+x(X_{pf}^2+X_{pt}^2) \right\}$ FOR $\begin{cases} Y_{pf}+Y_{pt} < 0, X_{pf} \neq X_{pt} < 0 \\ Y_{pf} \neq Y_{pt} > 0, X_{pf} \neq X_{pt} > 0 \end{cases}$
 FOR $e_y = 3.19$

SEGMENT ID	TRANSFORMED COORDINATES		C1		C2		$Z_x = C1 \times C2$
	Y_{pf}	Y_{pt}	A	$R+y$	$ R_{pf}+y_{pt} /2$	$(Y_{pf}^2+Y_{pt}^2)/2$	
05	-14.9288	-14.6254	2.184	-	14.7771	-	32.2731
07	-14.6254	1.263	-	0.376	-	107.7487	40.5115
09	1.263	1.2771	0.0002	-	1.27005	-	0.0002
10	1.2771	3.4215	1.6447	-	3.3493	-	3.8638
12	3.4215	3.4074	0.0002	-	3.4145	-	0.0007
13	3.4074	5.6016	0.825	-	4.5045	-	3.7162
14	5.6016	5.2982	2.184	-	5.4499	-	11.9027
21	-14.6254	-14.322	2.184	-	14.4737	-	31.6105
31	3.4215	3.9847	1.7154	-	3.7031	-	6.3525
33	3.9847	1.8404	0.8385	-	2.9126	-	3.4422
41	5.6016	5.905	2.184	-	5.7533	-	12.5653

$\Sigma Z_x = 145.2389$

† SPECIAL TREATMENT FOR $Y_{pf} = Y_{pt} = 0$ AND $X_{pf} = X_{pt} = 0$ IS OMITTED FOR IT IS NOT APPLICABLE IN THE CALC.

Given condition number (1): we can pseudo-finalize the "Segment Area Strategy" and apply a "mass-transformation" of the entire suite of the coordinates for every point of interest from the current $[Y_p]$ into $([Y_p] - [e_{yincr}])$. This step may be interpreted as a result that the Elastic Principal X-axis had been shifted deeper into the "positive Y zone" by the $[e_{yincr}]$ amount. Inevitably these transformed coordinates would bring changes to the area summation value(s) as well.

With perfect condition number (2): if as we had hoped for, the only numerical side effects from the preceding "mass coordinate transformation" would only apply to the value belonging to the $[\Sigma A_{+PART}]$ portion but not to the $[\Sigma A_{+FULL}]$.

To accommodate the pseudo-final incremental shift $[e_{yincr}]$, the affected aggregate area $[A_{+PART}]$ for each participating segment must now be

updated into a brand new entity $[A_{+PARTnew}]$. Accordingly from the pseudo codes, the formulation for that would be:

$$A_{+PARTnew} = ([Y_p] - [e_{yincr}]) * A / \Delta Y_p$$

In which for $Y_{pf} < 0$ and $Y_{pt} > 0$: $\Delta Y_p = (Y_{pt} - Y_{pf})$,
 and for $Y_{pf} > 0$ and $Y_{pt} < 0$: $\Delta Y_p = (Y_{pf} - Y_{pt})$,

Or simply: $\Delta Y_p = |Y_{pt} - Y_{pf}|$.

Equilibrium requires that $[(\text{existing } \Sigma A_{+FULL}) + (\text{new } \Sigma A_{+PARTnew})] = \Sigma A / 2$. By retaining the values for both $[\Sigma A_{+FULL}]$ and $[\Sigma A_{+PART}]$, which were "leftovers" from the iteration cycle immediately preceding

the current cycle, and treating $[e_{yincr}]$ as **unknown constant**, we can formulate, substitute, expand, rearrange, and solve for $[e_{yincr}]$ as follows:

$$\begin{aligned} \Sigma A_{+FULL} + \Sigma A_{+PARTnew} - \Sigma A / 2 &= 0 \\ \Sigma A_{+FULL} + \Sigma \{ ([Y_p] - [e_{yincr}]) * A / \Delta Y_p \} - \Sigma A / 2 &= 0 \\ \Sigma A_{+FULL} + \Sigma \{ [Y_p] * A / \Delta Y_p \} - [e_{yincr}] * \Sigma (A / \Delta Y_p) - \Sigma A / 2 &= 0 \\ \Sigma A_{+FULL} + \Sigma A_{+PART} - [e_{yincr}] * \Sigma (A / \Delta Y_p) - \Sigma A / 2 &= 0 \\ \Sigma A_{+FULL} + \Sigma A_{+PART} - \Sigma A / 2 &= [e_{yincr}] * \Sigma (A / \Delta Y_p) \end{aligned}$$

$$\text{solution: } [e_{yincr}] = (\Sigma A_{+FULL} + \Sigma A_{+PART} - \Sigma A / 2) / \Sigma (A / \Delta Y_p)$$

In which, appearing in the denominator, the discrete term $(A / \Delta Y_p)$ may be interpreted as the “Normalized segment area projected onto the $[Y_p]$ axis”. Very importantly, we must recognize that this term has nothing to do with $[A_{+FULL}]$ because it was derived straight from the expansion of $[A_{+PARTnew}]$. Therefore we need to keep in mind that this expression **applies only to segments with area straddling both zones**, or with a corresponding “segment area layout” evaluated as True to the condition of “partial $[+Y_p]$ and partial $[-Y_p]$ ”.

We had just finished elaborating the “**segment area strategy**” for bending about the X_p -axis. When applying this strategy as the basis for successive approximation, each succeeding $[e_{yincr}]$ would lead to:

- (1) new set of transformed coordinates $[Y_{pf}]$ and $[Y_{pt}]$ at segment incidences and
- (2) updated segment areas $[A_{+FULL}]$ and $[A_{+PARTnew}]$.

As a result, the new sum from adding $[\Sigma A_{+FULL}]$ to $[\Sigma A_{+PARTnew}]$ were checked against $[\Sigma A / 2]$ for convergence. The process would repeat as required until the conditions stipulated in “**Iteration Strategy**” were met.

A similar strategy can be formulated and derived for sections subjected to bending about the Y_p -axis as well. We would leave that as a homework exercise.

Convergence Strategy

When given a function $F(X)$ of variable ‘ X ’ and wish to solve for ‘ X ’ that satisfies $F(X) = X$, if a close-formed solution is not achievable then we could try solving it by “successive approximation”. It normally starts out with an initial approximation X_0 , substituting X_0 into $F(X)$ leads to the next trial value X_1 , then similarly the succeeding values $X_2, X_3 \dots$ and finally X_n at completion of n th iteration. The proceeding went something like this:

$$\begin{aligned} X_1 &= F(X_0) \\ X_2 &= F(X_1) \\ &\vdots \\ X_n &= F(X_{n-1}) \end{aligned}$$

The important question is: at which ‘ n ’ does the resulting $[X_n]$ converge to a prescribed precision threshold? Or asking bluntly: would it ever converge (or become diverging instead)? The answer could depend on many factors that can be looked up from literature on numerical analysis.

Generally if a solution scheme can lead to convergence at all, the required number of trials, ‘ n ’, had something to do with: (1) how close the initial guess $[X_0]$ was to the true solution value and (2) how stringent the convergence criterion was set.

But the main concern remains whether it would ever converge with any arbitrarily chosen $[X_0]$ as the initial trial value. Besides that, sometimes the winning odds in successive approximation are vastly influenced by the function’s first derivative $[F'(X)]$ and/or second derivative $[F''(X)]$. Therefore one of the prerequisites for convergence for most common equations is that the mathematical function is **differentiable**.

Our convergence measure calls for a simple validation: $[\Sigma A_{+FULL} + \Sigma A_{+PART} = \Sigma A / 2]$. Apparently our biggest problem being that this mathematical expression is neither a continuous function nor is it differentiable. However we could still solve it by “successive approximation” except with a minor “twist”. Let us start out from these self-explanatory terms:

$$\begin{aligned} \varepsilon &= \text{prescribed convergence tolerance} \\ A_{\text{sum}} &= \Sigma A_{+FULL} + \Sigma A_{+PART} \\ A_{\text{ideal}} &= \Sigma A / 2 \\ R_{\text{err}} &= \text{relative error ratio} = |(A_{\text{sum}} - A_{\text{ideal}})| / A_{\text{ideal}} \end{aligned}$$

Notice that $[A_{\text{ideal}}]$ is a constant. Already its value is known before the EC was located. Say, for bending about the X-axis, after a judiciously selection of:

- (a) initial $[e_{y\text{init}}]$ to start the first iteration cycle, or
- (b) a newly calculated value of $[e_{y\text{next}}]$ ($= [e_{y\text{init}}] + [e_{y\text{incr}}]$) prior to starting a brand new cycle,

we would obtain the area sum $[A_{\text{sum}}]$ per “Segment Area Strategy” followed by the relative error ratio $[R_{\text{err}}]$. Convergence is reached at the current cycle only when $R_{\text{err}} \leq \varepsilon$. Otherwise a new value of $[e_{y\text{incr}}]$ for the subsequent iteration cycle can be **estimated** as:

$$[e_{y\text{incr}}] = f_{\text{mod}} * [(\Sigma A_{+FULL} + \Sigma A_{+PART} - \Sigma A / 2) / \Sigma (A / \Delta Y_p)]$$

This is the same expression for $[e_{y\text{incr}}]$ derived under the “Segment Area Strategy” except for the extra term $[f_{\text{mod}}]$ in front. Intended for controlling the solution progress, $[f_{\text{mod}}]$ is a correction factor of real number, whose value is contingent on whether we wish to speed up or slow down the pace in reaching convergence.

Locating the PC for any generic section (whether symmetrical or not) is nothing but playing with geometry. To succeed we need to “play it right” at each and every iteration cycle, particularly with those segments “straddling an imaginary neutral axis”.

At conclusion of each cycle, the predecessor $[e_{y\text{init}}]$ or $[e_{y\text{next}}]$ would always lead to a succeeding $[e_{y\text{incr}}]$, whose value is determined by a fixed-formed algebraic formula. In most applications, modifying the succeeding $[e_{y\text{incr}}]$ by a factor $[f_{\text{mod}}]$ may be **unnecessary**. But there is something **interesting** in the numerical difference, or the **physical gap** (or jump), between the succeeding $[e_{y\text{incr}}]$ and the predecessor $[e_{y\text{next}}]$.

We may visualize the positive (or negative) gap as if the “axis of interest” had jumped forward (or backward). If the jump between **consecutive** cycles is relatively small then the respective “number” of segments participating in either $[\Sigma A_{+FULL}]$ or $[\Sigma A_{+PART}]$ would remain “unchanged”. By such “perfect condition”, it had proven the **truth** to the assumption statement: “**all the current A_{+FULL} stays being FULL and all the current A_{+PART} remains being PARTIAL**”. However this condition can become false if the gap created by the forward jumping is too excessive.

How excessive can be measured as too excessive is another question. But the fact is jumping **forward** too much can invariably add detrimental side effect to the succeeding $[e_{y\text{incr}}]$. Here is why: instead of jumping further forward, we could be trapped into exactly the opposite by reversing its course **mathematically** in the subsequent cycle and end up jumping backward. This could result into a situation that some of the predecessor “FULL” or “PARTIAL” segments would be “skipped over” due to this **unwanted** “numerical side effect”. Apparently those segments being “skipped over” would no longer contribute **properly** in the summation terms, $[\Sigma A_{+FULL}]$ and/or $[\Sigma A_{+PART}]$, for the duration of the succeeding cycle.

As the numerical back and forth (forward and reverse) compensation continues, it can result into a whole series of $[e_{y\text{incr}}]$ with elemental magnitudes **alternating** between values of its predecessor and successor. Much like initially that both X_1 and X_2 are unique and moderately far apart (numerically),

but recursively we encounter $X_3 = X_1$, $X_4 = X_2$, $X_5 = X_3$ and $X_6 = X_4$, etc., rendering an iteration process never converges. This situation could occur for:

- (1) section profiles with very **unusual** geometric segment setups and/or
- (2) formulation of $[e_{y_{incr}}]$ dictated by the unique feature of problem at hand.


One of the effective means to controlling convergence is through modification factor $[f_{mod}]$. A generalized modification can be applied in whichever ways that suit to global and/or local term(s) by addition and/or multiplication. To scale down the “jump” through multiplication in our situation, obviously $[f_{mod}]$ has to be less than 1. An important point to make here is that a factor too close to 1 would still carry unwanted risk of too much gap between successive cycles whereas a factor too small would definitely chalk up the number of ‘n’.

The concern is not how rapid it would converge but to have assurance of a reasonable solution (always). If we set it as the highest priority to minimize the chance of “not converging” then we should go for a smaller $[f_{mod}]$. Striking a balance in between, we have set the f_{mod} value to 0.5 in our example problem.

Locating the Plastic Principal Axes

Notice that some of the notations used in this article were slightly different from that appeared in the Calc. Now referring to Calc page 5 and page 6 on “iteration for X_p -axis bending”. We started out with the most convenient value of $e_{y_{init}} = 0$ at cycle #1 from that we obtained the next $e_{y_{incr}} = 3.1902$. It gives an error ratio $R_{err} = 0.1216$, which obviously is not acceptable.

Remember we made a statement: “In most applications, modifying the succeeding $[e_{y_{incr}}]$ by a factor $[f_{mod}]$ may be unnecessary”. To confirm if that is true, we applied $e_{y_{next}} = 3.1902$ without modification for the next cycle. “Yes indeed” it does converge immediately (not shown in the hand Calc).



Structural Design Corporation

Calc. For: PLASTIC SECTION MODULUS OF
W 21 x 55 + C 8 x 11.5

Calc. No. _____
Rev. _____ Date _____
Page 9 of 10

Client: _____
Project: SDC STANDARD

Prepared by: JOHN FONG Date: 9-13-11
Reviewed by: _____ Date: _____

FOR $e_x = -0.4979$

SEGMENT	TRANSFORMED COORDINATES		C1		C2		$Z_p = C_1 \times C_2$
ID	X_{pf}	X_{pt}	A	R_{ex}	$ X_{pt} + X_{pe} /2$	$(X_{pt}^2 + X_{pe}^2)/2$	
05	-3.5165	0.6126	2.184	0.5214	-	6.5833	3.4322
01	0.6126	-0.5382	9.9738	5.1912	-	0.3325	1.7257
09	-0.5382	-0.3437	0.002	-	0.4409	-	0.001
10	-0.3437	-0.499	1.6447	-	0.42135	-	0.693
12	-0.499	-0.6935	0.002	-	0.59625	-	0.001
13	-0.6935	-0.8524	0.825	-	0.77295	-	0.6377
14	-0.8524	-5.0414	2.184	-	2.9469	-	6.4361
21	0.6126	4.8016	2.184	-	2.7071	-	5.9123
31	-0.499	7.2781	1.7194	0.2206	-	26.6099	5.8696
33	7.2781	7.4334	0.8385	-	7.35975	-	6.1678
41	-0.8524	3.3366	2.184	0.5214	-	5.9297	3.0916

$\sum Z_p = 33.9661$

8. SHAPE FACTORS

$\begin{cases} \sum I_{xx} \\ \sum I_{yy} \\ \sum I_{xy} \end{cases} = \text{MOMENT OF INERTIA BASED ON USER SYSTEM (P.3)} = \begin{cases} 1314.7098 \\ 132.438 \\ 80.079 \end{cases}$

LET $D = [(\sum I_{xx} - \sum I_{yy})/4 + \sum I_{xy}^2] = 597.3703$

$\begin{cases} I_{xx}' \\ I_{yy}' \end{cases} = \text{PRINCIPAL ELASTIC AXIS BASED MOMENT OF INERTIA} = \begin{cases} +D \\ -D \end{cases} + \frac{1}{2}(\sum I_{xx} + \sum I_{yy}) = \begin{cases} 1320.9442 \\ 126.2036 \end{cases}$

$\begin{cases} y' \\ x' \end{cases}_{max/min} = \text{EXTREME FIBER COORDINATE FOR } \begin{cases} x' \\ y' \end{cases} \text{ BENDING} = \begin{cases} 11.7387 \\ 6.9355 \end{cases} \text{ @ } \text{NOTE } \begin{cases} 0.4 \\ 3.7 \end{cases} = \begin{cases} 9.0951 \\ 5.6394 \end{cases}_{min}$

That should be a perfect ending for such an accurate resolution after only two tries without bothering with $[f_{mod}]$. But we decided to carry on **purposely** with $f_{mod} = 0.5$ and observe what it takes if we only shoot for an acceptable error ratio somewhere below 0.1%.

As shown on Calc page 6 that continued from $e_{y_{next}} = 1.5951 (= 0.5 * 3.1902)$ at cycle #2, immediately the R_{err} dropped to 0.0304 or at 3% error. As expected at cycle #5, the R_{err} reduced further to 0.0076 giving next $e_{y_{next}} = 3.0904$ that is fairly close to $e_{y_{solution}} = 3.1902$ already. Finally we carried on up to cycle #8 and arrived at $e_{y_{next}} = 3.1777$ with an error ratio $R_{err} = 0.00095$, which is trivial enough to stop further trials.

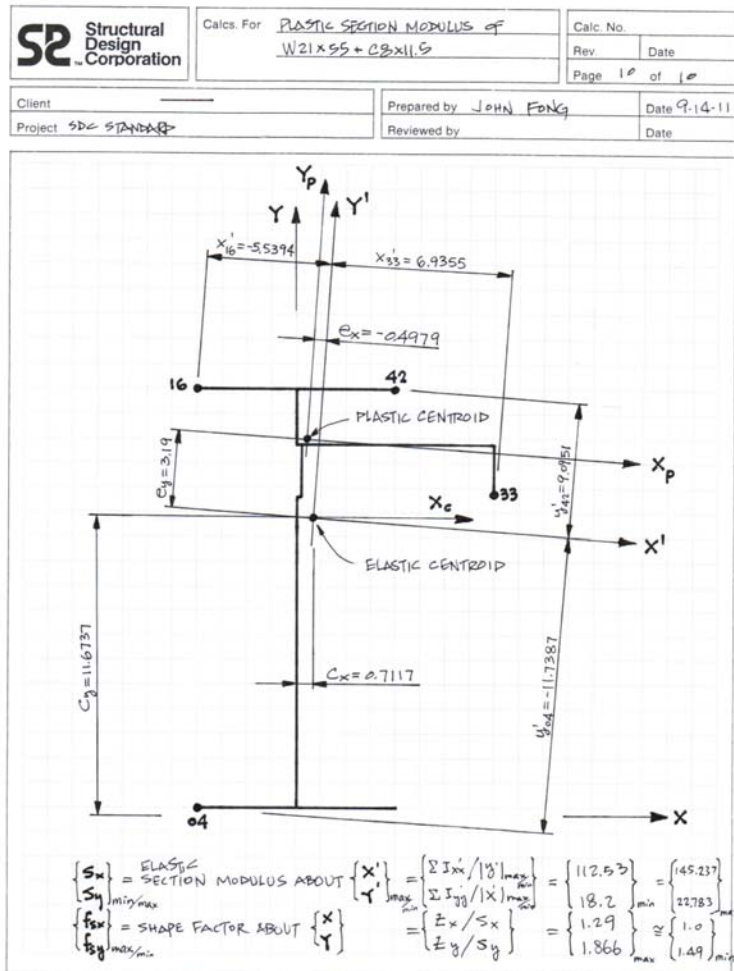
Similarly we initiated the “iteration for Y_p -axis bending” from $e_{x_{init}} = 0$ and arrive at $e_{x_{next}} \approx -0.49$ after 6 cycles with an error ratio $R_{err} = 0.0102$ (see Calc page 7).

Obviously the number of iterations required for X_p -axis bending and that for Y_p -axis bending could be the same or different (as our example showed). Other than that it is really our call when to quit the process by reviewing the error ratio $[R_{err}]$ against a prescribed threshold. Here we had just proven a few specifics regarding the required number of trials, ‘n’:

- (1) it may be highly dependent on what f_{mod} is and how it fits in the formulation
- (2) it decreases when e_{next} is set closer to $e_{solution}$
- (3) it increases if we set the prescribed convergence tolerance too small.

The generic strategy used in our example can be summarized into these major steps:

- (1) start from $[e_{init}] = 0$ at cycle #1 and obtain $[e_{incr}]$. Stop process if converging otherwise continue to step (2)
- (2) without modification (by letting $[f_{mod}] = 1$), try $[e_{next}]$ as full value of $[e_{incr}]$ at cycle #2. Stop if converging otherwise continue to step (3)
- (3) Try $[e_{next}] = [e_{previous}] + 0.5 * [e_{y_{incr}}]$ for all the subsequent cycles until converging. But that may or may not be the most favorable setup yet. In some occasions using the “interval



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halving method” with $[e_{next}] = 0.5 * ([e_{previous}] + [e_{yincr.]})$ may work better for certain profile geometry.

Plastic Section Modulus

PSM, by definition, is the summation of aggregate “static moments of area” applied to all the participating components about the respective **PPA**. The engineering term “Static moment of area” (**SMA**) is synonym to “Static Moment”, “First Moment of Area” or simply the “First Moment”. Even though the **SMA**’s mathematic formulation for **PSM** is similar to that for the **SMA** involved in calculating the flexural horizontal shear stress (**HS**), but they are different in several ways:

- PSM** is a unique constant for the entire profile. It is an overall sum of **SMA** from all the components in a profile while **SMA**s for **HS** vary from node to node and is calculated as the accumulation (numerical integration) of **SMA** from a terminal node up to the local point of interest,
- PSM** is a pure scalar that takes no notice of the (+/-) sign carried by the coordinates in whichever quadrant while **SMA** for **HS**, in a way a nodal vector representing qualitatively the flexural shear flow, demands the otherwise.

Symbols $[Z_x]$ and $[Z_y]$ are normally assigned to correspond with **PSM** for bending about the **PPA** $[X_p]$ and $[Y_p]$ axes.

With reference to X_p -axis bending: before starting the $[Z_x]$ calculation, it is essential to declare an initial condition: $Z_x = 0$. The **pseudo codes**, for a typical segment, would work something like this:

Let the “full area” of the current segment = A
 Retrieve coordinates Y_{pf} and Y_{pt} (by coordinate transformation based on offset e_{y_final})
 Let $\Delta Y = |Y_{pf} - Y_{pt}|$

Start evaluating coordinate Y_{pf}

$Y_{pf} < 0$
 If only $Y_{pt} > 0$ then
 $Z_x = Z_x + 0.5 * (Y_{pt}^2 + Y_{pf}^2) * (A / \Delta Y)$

Otherwise
 $Z_x = Z_x - 0.5 * A / (Y_{pf} + Y_{pt})$

$Y_{pf} = 0$
 If only $Y_{pt} > 0$ then
 $Z_x = Z_x + 0.5 * A * Y_{pt}$
 If only $Y_{pt} = 0$ then
 $Z_x = Z_x + 0.25 * A^2 / \Delta Y$

If only $Y_{pt} < 0$ then
 $Z_x = Z_x - 0.5 * A * Y_{pt}$

$Y_{pf} > 0$
 If only $Y_{pt} \geq 0$ then
 $Z_x = Z_x + 0.5 * A * (Y_{pf} + Y_{pt})$
 If only $Y_{pt} < 0$ then
 $Z_x = Z_x + 0.5 * (Y_{pt}^2 + Y_{pf}^2) * (A / \Delta Y)$

End of Y_{pf} evaluation

The final sum of **PSM** Z_x is realized after applying the $[Y_{pr}]$ evaluation for each and every segment. As an exercise, the pseudo codes for **PSM** Z_y bending about the Y_p -axis could also be derived using similar coding strategy.

Now we are ready to calculate the **PSMs** after finalizing the **PPA** axis offsets from the **EPA** ($e_{x_final} = 3.19$ and $e_{y_final} = -0.4979$). **Calc** pages 8 and 9 summarize the process of how Z_x and Z_y were calculated for our example problem.

Shape Factors

“Form factor” is a term appears akin to “Shape factor”. But they are nothing like each other, at least in the engineering mechanics relevance. Both terms have diverse meaning in diverse genres of application. Depending on what it refers to, even “Form factor” can be significantly different in material stress-related subjects. However “Form factor” is not our concern here for it has nothing to do with “Plastic Center” or “Plastic Section Modulus”. We point it out to preclude any misunderstanding if there is.

Let us now concentrate on the Shape Factor (**SF**). For bending about either the strong axis or the weak axis, if we let M_y = yield moment and M_p = plastic moment, then **SF** is known as M_p / M_y . By simple arithmetic, it deduces to the ratio of **PSM** / **ESM** where **PSM** is based on $[X_p]$ and $[Y_p]$ per **PPS** while **ESM** is rooted from $[X']$ and $[Y']$ per **EPS**.

Numerical treatment specific to **SF** requires the smallest value of **ESM**. They were evaluated for either axis of bending at the extreme fibers or at nodes located furthest away from the **EC** (along both positive and negative senses).

Referring back from **Calc** page 3 for X' -bending, we would look for the maximum value: $[Y'_{max}]$ and the minimum value: $[Y'_{min}]$ under the Y_i' and Y_j' columns. Likewise for Y' -bending, we would find the maximum value: $[X'_{max}]$ and the minimum value: $[X'_{min}]$ under the X_i' and X_j' columns, respectively.

Calculation of **ESM** at any nodal point requires gross principal moments of inertia $[\Sigma I'_{xx}]$ and $[\Sigma I'_{yy}]$ formulated as:

$$\Sigma I'_{xx} = 0.5 * (\Sigma I_{xx} + \Sigma I_{yy}) + [0.25 * (\Sigma I_{xx} - \Sigma I_{yy})^2 + \Sigma I_{xy}^2]$$

$$\Sigma I'_{yy} = 0.5 * (\Sigma I_{xx} + \Sigma I_{yy}) - [0.25 * (\Sigma I_{xx} - \Sigma I_{yy})^2 + \Sigma I_{xy}^2]$$

Also refer to **Calc** pages 9 and 10, it follows that the minimum and maximum **ESM** for bending about both axes are:

$$\{ESM_{x_min}, ESM_{x_max}\} = \{\Sigma I'_{xx} / Y'_{max}, \Sigma I'_{xx} / Y'_{min}\}$$

$$\{ESM_{y_min}, ESM_{y_max}\} = \{\Sigma I'_{yy} / X'_{max}, \Sigma I'_{yy} / X'_{min}\}$$

Finally the shape factors, for the strong axis and weak axis bending, $[SF_x]$ and $[SF_y]$:

$$SF_x = \{Z_x / ESM_{x_min}, Z_x / ESM_{x_max}\} = \{1.29, 1.0\}$$

$$SF_y = \{Z_y / ESM_{y_min}, Z_y / ESM_{y_max}\} = \{1.866, 1.49\}$$

For bending about each respective axis, we calculated two **SFs** based on the extreme fiber located in both the positive and negative quadrants. One of the values represents yielding in tension and the other represents yielding under compression.

Except for very short and stubby members, in normal practices compression is almost always much more critical. All because, prior to general yielding, buckling can be triggered at a much lower stress level due to material or construction imperfections, load eccentricity, lack of support for compression fiber against lateral movements or excessive slenderness in either global and/or local

element, and so on. In addition, for profile sections subjected to generalized loading/force resulting from all six degrees of freedom, local buckling due to compressive fiber stress can be set off not only by bending stress, but also with combination from warping normal stress. That is why compression should always be treated with extra care.

For thin-walled sections, evaluating the **global** implication from shape factors is as important as checking for **local** buckling per component compact criteria. In our example problem for Y'-axis bending, the smaller 1.49 SF_y value can be interpreted as the Plastic Moment strength $[M_{pv}]$ that bent about the principal weak axis is at about 49% greater than the Yield Moment strength $[M_{yy}]$. In other words it would take that much more beyond yielding for the section to become plastified from weak axis bending. But on the other hand there could be a problem pertaining bending about the strong principal axis because the smaller 1.0 SF_x clearly indicated that soon after the extreme fiber reaches yield, the entire section could go into **plastification** immediately with very little (or no) margin left in between under the worst-case scenario. Therefore on the safe side we should always pay attention to whichever the smaller value that governs.

Conclusion

For our example profile, if the iteration process for either axis of bending takes only three (3) cycles to locate the PC and then calculate the PSM, we could probably have to deal with 1000+ numeric entries plus mathematical operators, parentheses and sign change keys, etc. Number crunching in successive approximation and meeting convergence criteria are quite different from plugging values to simple formulas involving limited number of parameters or constants. What is against us the most in this case should be the numerical errors accumulated from rounding and truncation. Even by punching the same number of digits (with double precisions) the same way into different tools (whether calculators, spreadsheets or database tables/queries), we should not anticipate to reproduce identical results from one Calc to the other.

Anyhow it should be much easier to go through the actual Calc than following the tedious narrative given in the article. But are we done yet? The answer may be “Yes” or “No” depending on what the application is for.

Keep in mind that the algorithm is only an **approximation** for **PSM** of **thin-walled sections**. Although the strategy presented here were verified extensively and it worked for many profile cases but there is no guarantee that it's bug free. For that our answer is “yes we are done” (or uhm may be).

The purpose of the pseudo code listings is only to delineate all the practical scenarios through logical branching statement blocks. However no algorithm is ever complete and free from further enhancement. If we opt to try this on **thick-walled sections** then by all means our answer would be “No, we are not done”. For that the algorithm must be expanded in greater details especially in handling a few more special conditions, among them: (1) how to handle inaccuracy resulting from overlapping of area at joints where multiple segments meet due to generic inadequacy from simplified skeleton middle line model, (2) probable non-convergence due to unusual arrangement of segments for some unusually-shaped profile and (3) what happens when $X_{pf} = X_{pt} \neq 0$ and $Y_{pf} = Y_{pt} \neq 0$. (more fun for another interesting homework problem.)